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# FIRM VALUATION, CORPORATE TAXES, AND DEFAULT RISK

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## I. INTRODUCTION

SINCE MODIGLIANI AND MILLER [10] demonstrated that the values of firms in the same risk class are equal and are independent of the capital structure of the firm if the probability of default is zero and there are no taxes, a number of extensions of these results to the case of default risk have been given by Stiglitz [16], Smith [15], Baron [2, 3], and Hagen [5]. Conditions under which the values of firms in a risk class will be equal when there is default risk are given in [2]. While the relative values of firms in a risk class are equal, that common value of the firms is not independent of its financing, in general. If changes in the capital structure do not alter the available returns in the capital market, the capital market is complete and the value of the firm will be independent of its capital structure as indicated in [3]. This will be true if investors are able to create homemade leverage as considered by Stiglitz and Baron and thus duplicate any pattern of returns the firm can create by issuing bonds. The value of a firm is also independent of its capital structure in mean-variance models (for example [9, 12, 14]) and in an Arrow-Debreu model (see [8, 16]). Smith has considered the important issue of an investor's preferences for the financing of a firm with a variable scale in the presence of default risk and has demonstrated that investors are not in general indifferent to the financing of a firm with a variable scale in the presence of default risk when the capital market is incomplete. A synthesis of the results regarding the effects of capital structure is given in [3].

A corporate profits tax causes the value of a firm to depend on its debt-equity ratio even with a complete capital market. In the absence of default risk, Modigliani and Miller [11] determined that the value of a levered firm exceeds the value of an unlevered firm in the same risk class by the present value of the tax savings created by the debt in the capital structure. Kraus and Litzenberger [8] have obtained the same result in a contingent claims model. The objective of this paper is to explore the effect of corporate income taxes in the presence of default risk on the value of firms in the same risk class when the financing of the firm does not alter the available returns in the capital market. The analysis is in the spirit of the original works of Modigliani and Miller in the sense that the results are obtained from a partial equilibrium analysis. The analysis is based on a model that makes few restrictive assumptions regarding the probability distribution of earnings of firms or the preferences of inves-

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tors. With these weak assumptions, however, exact relationships between the values of firms will not be obtained. Instead, to indicate the effects of debt financing, bounds on the relative values of levered and unlevered firms will be determined, and the bounds will be analyzed in terms of the default risk, tax rates, and the debt level. Two methodologies will be used to develop bounds on the values of firms. In sections IIA-IIC, alternative investments in the securities of the levered and unlevered firms are considered such that the returns per dollar invested are identical and hence the values of the investments will be equal. In sections IID-IIF, investors who only make investments in the securities of the firms in a risk class and in a risk-free asset are considered using stochastic dominance.

## II. Investor Returns and Relative Values

### A. *Returns on Levered and Unlevered Firms*

Consider two firms, or the same firm before and after debt financing, with identical probability distributions of gross, before taxes and financial charges, earnings X such that in any state of nature that occurs both firms have the same earnings.[1] For the purpose at hand, it suffices to consider a two-period model in which investors make portfolio decisions, the state occurs determining the earnings and the return to investors, and then the firms cease to exist. Firm 1 is assumed to be financed solely by equity, while firm 2 is financed by both debt and equity. The market value $V_1$ of firm 1 equals the value $E_1$ of its equity, while the market value $V_2$ of firm 2 equals the market value of its equity $E_2$ plus the value $D_2$ of its debt. Debt is assumed to sell at its par value and to carry a gross, nominal interest rate of r ($r > 1$). Investors are assumed to have preferences over their after-tax receipts, and for the present, no assumption will be made on the preferences except that a greater return is preferred to a lesser return. A risk-free gross interest rate $r^\circ$ is assumed to be exogenously given, and transactions costs will be assumed to be zero.

To develop the returns on the securities of the firms, consider an investor who holds an $\alpha$ proportion of the equity of the unlevered firm. The total return to all shareholders is zero if the earnings X are less than or equal to zero, and for $X > 0$ the investor receives a share of the after-tax profits $X(1 - T)$, where T is the corporate profits tax rate. The distribution of firm profits to investors is thus[2]

$$Y_1 = \begin{cases} 0 & \text{if } X \le 0 \\ X(1 - T) & \text{if } X \ge 0. \end{cases} \tag{1}$$

The return on the equity and bonds of the levered firm depends on the tax treatment of the bond principal and interest. The tax structure to be

1. More generally, the earnings of the firm may differ by a positive multiplicative constant in each state..

2. The returns defined in (1), (2), and (3) below are based on the assumption that there is no capital gains tax and no personal income taxes. Consequently, the return of capital need not be considered.

considered is that which most closely represents, in the context of a two-period model, the U.S. tax code in which interest charges are tax deductible but principal payments are not. If the gross earnings of the levered firm are such that there is no default on bond principal and interest, the tax is assessed on before-tax profit $(X - (r - 1)D_2)$. The distribution $R_E$ of after-tax profit to the equity owners is thus

$$R_E = \begin{cases} 0 & \text{if } X \leq D_2/(1 - T) + (r - 1)D_2 \equiv X^\circ \\ (X - (r - 1)D_2)(1 - T) - D_2 & \text{if } X \geq X^\circ, \end{cases} \tag{2}$$

where $X^\circ$ is the level of earnings that exactly covers the bond principal and interest. If the gross earnings are less than $X^\circ$, the firm defaults and the after-tax earnings accrue to the bondholders. If $X \leq (r - 1)\, D_2$, the firm defaults on the interest charges, and no tax is incurred. If earnings exceed the interest charges but do not exceed $X^\circ$, the tax is assessed on the earnings in excess of the interest charges. Consequently, repayment of bond principal and equity investment is made from after-tax profits.[3] The gross return $R_D$ to bondholders is

$$R_D = \begin{cases} 0 & \text{if } X \leq 0 \\ X & \text{if } 0 \leq X \leq (r - 1)D_2 \\ (X - (r - 1)D_2)(1 - T) + (r - 1)D_2 & \text{if } (r - 1)D_2 \leq X \leq X^\circ \\ rD_2 & \text{if } X \geq X^\circ. \end{cases} \tag{3}$$

## B. *A Lower Bound on the Valuation Effect of Debt Financing*

The analysis of the relative values of the levered and unlevered firms will initially be based on the result that two investments with identical returns in each state of nature will have the same value. In order to develop a lower bound on the relative values, consider an investment in an $\alpha$ proportion of the equity of the unlevered firm. Suppose that the investor considers selling that investment for $\alpha V_1$ and purchasing equity and debt of the levered firm in the ratio $(E_2/Q_2)$ and $kD_2/Q_2$, where $Q_2 = E_2 + kD_2$ and $k = ((r - 1)(1 - T) + 1)/r$. Then the investor owns

3. This tax structure is one of two basic structures that have been considered in the literature. The difference between the two structures, in the context of a two-period model, relates to the tax deductibility of bond principal. Modigliani and Miller [11], Kraus and Litzenberger [8], and Rubinstein [14], for example, assume that the bond principal is deductible which is in the spirit of an infinite horizon model. The tax structure considered here assumes only that the interest charges, and not the debt principal, are tax deductible. This seems more reasonable in the context of a two-period model and more realistic in view of the U.S. tax code (see also [17]). Another reason for not assuming that the debt principal is deductible is that the issue of firm valuation in the face of default risk is trivial in that case. With principal and interest deductible the total return on the equity of the levered firm is 0 if $X \leq rD_2$ and is $(X - rD_2)(1 - T)$ if $X \geq rD_2$ while the total return on bonds is X if $X \leq rD_2$ and is $rD_2$ if $X \geq rD_2$. Then, a $(1 - T)\alpha$ share of the bonds and a $\alpha$ share of the equity of a levered firm yields the same return for all values of X as an $\alpha$ share of the equity of the levered firm. The values of the firms thus are related by $V_2 = V_1 + TD_2$. This is, for example, the linearity property that permits Rubinstein to obtain the Modigliani-Miller result in a mean-variance model. His method of proof cannot be used for the tax structure considered herein, if there is a risk of default on the interest payment, since the return on the unlevered firm cannot be expressed as a linear combination of the return on the debt and equity of the levered firm. If there is no risk of default on the interest payments with the tax structure considered here, the return on the unlevered firm equals the return of the debt plus the equity of the levered firm minus the quantity $T(r - 1)D_2$. This will be used below to develop an exact relationship between the values of the firms in the absence of default risk on the interest charges and a bound on the values in the presence of such a risk.

$\alpha(V_1/Q_2)E_2$ of the equity and $\alpha(V_1/Q_2)kD_2$ of the debt of the levered firm, and the gross return $Y_2$ is

$$Y_2 = \alpha\begin{cases} 0 & \text{if } X \le 0 \\ (V_1/Q_2)kX & \text{if } 0 \le X \le (r-1)D_2 \\ (V_1/Q_2)k(X(1-T) + T(r-1)D_2) & \text{if } (r-1)D_2 \le X < X^\circ \\ (V_1/Q_2)X(1-T) & \text{if } X \ge X^\circ. \end{cases} \tag{4}$$

If all investors believe that there is no default risk ($X \ge X^\circ$ with probability one), the nominal interest rate r equals the risk-free rate $r^\circ$. The returns $Y_1$ and $Y_2$ are then identical when $V_1 = Q_2$, which implies that in equilibrium

$$V_1 = Q_2 = E_2 + kD_2 = V_2 - (1-k)D_2 = V_2 - TD_2(r-1)/r. \tag{5}$$

The condition in (5) implies that the returns per dollar invested in each of the firms are equated. The return per dollar invested in the unlevered firm is $X(1-T)/V_1$ and for the levered firm is $X(1-T)/Q_2$, so if, for example, $V_1 > Q_2$ every investor prefers to sell equity of the unlevered firm and purchase bonds and equity of the levered firm.[4] The value of the levered firm thus exceeds the value of the unlevered firm by the present value of the tax savings ($TD_2(r-1)$) created by the tax-deductibility of the interest charges, where the discount factor is r which equals the risk-free rate $r^\circ$. The result in (5) is a variant of the Modigliani-Miller tax result (11, Eq. (3)).[5] Consequently, if all investors believe that $X_{\min}$ is the minimum earnings of the firm, the firm may issue debt until $D_2 = X_{\min}(1/(1-T) + r^\circ - 1)$ and the value of the firm will increase at a constant rate of $T(r^\circ - 1)/r^\circ$ per dollar of debt issued.

The returns in (1) and (4) may be used to develop a lower bound on the relative values of levered and unlevered firms for any level of debt. The return $Y_2$ in (4) may be rewritten as

$$Y_2 = (V_1/Q_2)Y_1 + \alpha\begin{cases} 0 & \text{if } 0 \le X \\ ((V_1/Q_2)k - (1-T))X & \text{if } 0 \le X \le (r-1)D_2 \\ ((V_1/Q_2)k - 1)X(1-T) & \\ \quad + (V_1/Q_2)kT(r-1)D_2 & \text{if } (r-1)D_2 \le X \le X^\circ \\ 0 & \text{if } X \ge X^\circ. \end{cases} \tag{6}$$

If $V_1 = Q_2$, then $((V_1/Q_2)k - (1-T)) = T/r$, so $Y_2 > Y_1$ if $X \epsilon (0, (r-1)D_2)$. For $X \epsilon [(r-1)D_2, X^\circ]$ and $V_1 = Q_2$, $Y_2 = Y_1 + T(r-1)(kD_2 - X(1-T))$, where the last term is positive. Consequently, in the presence of default risk the return $Y_2$ is at least as great as $Y_1$ in every state of nature if $V_1/Q_2 = 1$, and hence the return per dollar invested is at least as great in each state for the levered firm as for the unlevered firm. The value of $Q_2$ thus is at least as great as $V_1$,[6] and the relationship in (5)

4. The process that will yield this result in the absence of default risk is the same as that considered in [2] and [16]. For example, if $V_1 > Q_2$, all equity holders prefer to sell their equity and to purchase bonds and equity of the levered firm, so $V_1 \not> Q_2$. If $Q_2 > V_1$, all investors may create homemade leverage and sell equity in the levered firm and buy equity in the unlevered firm until $Q_2 \not> V_1$. Consequently, $Q_2 = V_1$.

5. Modigliani and Miller state the result as $V_1 = V_2 - TD_2$.

6. That is, if $V_1 = Q_2$ and all investors believe that there is a probability of default, the return on a

is a lower bound for all levels of debt financing and for all probability distributions of X.

## C. *An Upper Bound on the Value of Debt Financing*

An upper bound on the effect of debt financing may be obtained by considering a different pair of investments in the securities of the levered and the unlevered firms. All investors will now be assumed to believe that there is a zero probability of default on interest payments. Consider the same investment in an $\alpha$ share of the unlevered firm but an $\alpha$ share of the bonds and equity of the levered firm. For the case in which there is no risk of default on the interest charges but there is a risk of default on the debt principal, the gross return on the first investment if $Y_1$ given in (1) and on the second is

$$Y^*_2 \equiv \alpha R_D + \alpha R_E = Y_1 + \alpha T(r - 1)D_2 \qquad \text{if } X \geq (r - 1)D_2. \tag{7}$$

Whatever state of nature occurs, the return from the second investment yields $\alpha T(r - 1)D_2$ more than the first investment, so the value of the second investment must exceed the value of the first by the value of the future return $\alpha T(r - 1) D_2$. Consequently, if the probability of default on the interest payments is zero,[7]

$$V_2 = V_1 + T(r - 1)D_2/r^\circ, \tag{8}$$

which is analogous to the original Modigliani-Miller result for the tax structure given here. The condition in (8) indicates that an $\alpha$ share of the bonds and equity of the levered firm is identical to an $\alpha$ share of the equity of the unlevered firm plus savings equal to $\alpha T(r - 1)D$, since the return per dollar invested is the same in both cases.

If there is no risk of default on the bond principal, the nominal interest rate

portfolio including $Y_2$ dominates the return on a portfolio including $Y_1$ in the place of $Y_2$. In a partial equilibrium sense, all investors prefer to sell equity in the unlevered firm and to purchase equity and bonds of the levered firm, which will raise the value of the levered firm relative to the value of the unlevered firm.

7. A more formal demonstration of the condition (8) may be given if any one of the following conditions holds: (a) all investors hold bonds of firms in the same risk class, (b) investors can borrow at the risk-free rate when there is no risk to the lender, or (c) investors can sell short. The demonstration involves showing that identical returns can be obtained by investing in either the levered or the unlevered firm. Condition (7) demonstrates that the returns on the unlevered and levered firms are related by a linear function, so it is only necessary to demonstrate that such a linear function may be formed from the available securities in the market. First, consider the owners of the unlevered firm. They may obtain the return $Y^*_2$ by selling their equity in the unlevered firm and reducing their savings by $\alpha T(r - 1)D_2$, or if savings are not that great, by borrowing at the risk-free interest rate by pledging an equal amount of the certain portion ($T(r - 1)D_2$) of the earnings of the levered firm. The returns on these investments are then identical. Alternatively, an investor could purchase an $\alpha$ proportion of the debt and equity of the levered firm by selling short $\alpha V_1$ of the equity of the unlevered firm and borrowing $\alpha T(r - 1)D_2$ pledging the certain portion of the return as collateral. The gain on this transaction is zero in every state of nature. Next, the equity owners of the levered firm may invest $\alpha V_1$ in the equity of the unlevered firm and save $\alpha T(r - 1)D_2/r^\circ$ by 1) selling their equity ($\alpha E_2$) and selling ($\alpha D_2$) of bonds of the levered firm, or 2) selling their equity and selling short sufficient bonds ($\alpha D_2$), or 3) selling short $\alpha E_2$ of equity and $\alpha D_2$ of debt of the levered firm. Instead of presenting the proofs of these statements at this point, the demonstrations will be given below for the more general case in which there is a risk of default on the interest charges.

r equals $r^\circ$, and the result in (5) obtains. The more general expression in (8) may be rewritten as

$$V_2 = V_1 + \frac{T(r^\circ - 1)D_2}{r^\circ} + \frac{T(r - r^\circ)D_2}{r^\circ}.$$

In the presence of default risk on the bond principal the last term is positive representing an additional value to the firm from a level of debt financing that results in a risk of default and hence a nominal interest rate greater than $r^\circ$. The relationship in (8) may also be rewritten as

$$V_2 = V_1 + T(r - 1)D_2/r + (T(r - 1)D_2/r)(r - r^\circ)/r^\circ, \tag{9}$$

where the first two terms on the right-side are those given in (5). The lower bound in (5) is thus exceeded by the last term in (9) which represents the additional increase in value due to increasing the debt level to a point at which there is default risk on the bond principal.

In the presence of default risk on the payment of interest, the gross return from an $\alpha$ proportion of the bonds and equity of the levered firm may be written as

$$Y_2 = Y_1 + \alpha\begin{cases}0 & \text{if } 0 \leq X \\ TX & \text{if } 0 \leq X \leq (r - 1)D_2 \\ T(r - 1)D_2 & \text{if } X \geq (r - 1)D_2.\end{cases} \tag{10}$$

The second term on the right of (10) is less than or equal to $T(r - 1)D_2$ for all values of X such that there is a risk of default on interest payments, so the value of the levered firm may not exceed the value of the unlevered firm by more than $T(r - 1)D_2/r^\circ$. The expression in (8) is thus an upper bound on the relationship between the values of the levered and unlevered firms for all debt levels and all probability distributions of X. Using the lower bound in (5), the difference in firm values is bounded by

$$\frac{T(r - 1)D_2}{r^\circ} \geq V_2 - V_1 \geq \frac{T(r - 1)D_2}{r}. \tag{11}$$

Both inequalities hold as equalities with $r = r^\circ$ if there is no default risk, while the first inequality holds as an equality and the second as a strict inequality if there is a risk of default on the bond principal but not on the interest payments.

The upper and lower bounds given in (11) provide empirically testable hypotheses for firms in a risk class, since both the nominal and the risk-free interest rates are observable. The difference in the bounds is

$$T(r - 1)D_2(r - r^\circ)/(rr^\circ),$$

which is rather small per unit of debt. For example, if $T = 0.48$, $r = 1.1$, and $r^\circ = 1.08$, this difference is 0.000808 per dollar of debt. The rate of increase in the value of the firm as debt is increased is between 0.0436 and 0.0444. The maximum difference in the values of the firms occurs when earnings are certain at a value of $X^\circ = (r - 1)D_2$ and debt is increased to that level, so that all the earnings of the levered firm are protected from taxes. The value

of the equity of the levered firm is then zero, and hence, $V_2 = D_2$. The nominal interest rate on the debt must be such that the return $(r - 1)D_2$ equals the return on the risk-free asset or $(r - 1)D_2 = r^\circ D_2$ which implies that $r = r^\circ + 1$. Then, the value of the unlevered firm is $V_1 = (r - 1)D_2(1 - T)/r^\circ = D_2(1 - T)$, and

$$V_2 = V_1 + TD_2,$$

which is identical to the upper bound in (8). Such a debt level is, of course, prohibited by the tax authorities, who would in all likelihood declare the bonds to be shares thus eliminating the tax benefits. The firm however may be able to increase the debt level to the point at which the probability of default on the interest charges is positive, and all value-maximizing firms have an incentive to do this. To study the effect of further increases in debt, the concept of second-degree stochastic dominance will be utilized.

## D. *Firm Values in the Presence of Default Risk on Interest Charges*

An upper bound on the difference in the values of the firms may be developed for the case in which there is a risk of default on interest payments and investors have homogeneous expectations. Consider an investor who holds an $\alpha$ proportion of the bonds and equity of the levered firm and is considering selling those securities for $\alpha V_2$, purchasing $\alpha V_1$ of the equity of the unlevered firm, and saving the difference $\alpha(V_2 - V_1)$. The return on the levered firm is given in (10) while the return $Y^*_1$ on the second investment is

$$Y^*_1 = \alpha \begin{cases} (V_2 - V_1)r^\circ & \text{if } X \leq 0 \\ X(1 - T) + (V_2 - V_1)r^\circ & \text{if } 0 \leq X. \end{cases} \tag{12}$$

From the upper bound in (8) $V_2 - V_1 \leq T(r - 1)D_2/r^\circ$, so if this inequality is strict, the return in (10) on the levered firm is greater than $Y^*_1$ for $X > (V_2 - V_1)r^\circ/T$, but for $X < (V_2 - V_1)r^\circ/(1 - T)$, $Y^*_1$ is the larger. The return $Y^*_1$ is thus less likely to result in a very low return, since the difference in values of the firms is saved at the risk-free interest rate. That is, the return $Y^*_1$ is always positive while $Y_2$ may be zero.

Consequently, when there is a risk of default on the interest payments, it is not in general possible to form a linear combination of the securities of the levered firm that yields the same return as that in the unlevered firm. The analysis that will be used for this case is based on a special case of second-degree stochastic dominance [4, 7].[8] The distribution functions of $Y^*_1$ and $Y_2$ are such that they cross only once and the distribution function of $Y^*_1$ is less (greater) than that for $Y_2$ below (above) the crossing point. The return $Y^*_1$ thus dominates $Y_2$ for all concave utility functions if and only if the mean $\mu^*_1$ of $Y^*_1$ is at least as great as the corresponding mean $\mu_2$ for $Y_2$ (see [7, Theorem 3]). All investors will therefore wish to exchange $Y_2$ for

8. The analysis here is for an investor who has a portfolio consisting only of the securities of the firms in the risk class plus the risk-free asset or other risky assets whose returns are independent of X.

$Y^*_1$ by selling securities of the levered firm, purchasing equity of the levered firm, and saving at least until $\mu_2 \geq \mu^*_1$. This inequality may be evaluated as

$$\mu_2 - \mu^*_1 = -T \int_0^{(r-1)D_2} F(X)dX + T(r-1)D_2 + (V_2 - V_1)r^\circ \geq 0,$$

which implies

$$V_2 - V_1 \leq \frac{T(r-1)D_2}{r^\circ} - \frac{T}{r^\circ} \int_0^{(r-1)D_2} F(X)dX. \tag{13}$$

The upper bound in (8) is tightened by the last term in (13) if all investors have concave utility functions and hold bonds as well as equity of the levered firm. The last term represents the decrease in the value of debt financing due to the risk that all the tax deduction will not be utilized if $X < (r-1)D_2$. If the tax deduction will be fully utilized with probability one, the last term in (13) is zero and (8) is obtained.

As an example, let F(X) be a distribution function defined by

$$F(X) = X/(10(r-1)D) \text{ for } X \epsilon [0, (10(r-1)D)].$$

There is thus a probability of 0.10 that all of the tax shield will not be utilized and (13) becomes

$$V_2 - V_1 < \frac{19}{20} \frac{T(r-1)D}{r^\circ}.$$

The upper bound is thus 5% tighter than that given in (8) with the difference representing the loss in value of the levered firm due to the possibility that the tax shield will not be fully utilized.

The effect of debt financing on the value of a firm may be determined by differentiating, with respect to $D_2$, the bound on the value of debt financing given by the right side of (13), which will be denoted by $V(D_2)$. The derivative is[9]

$$\frac{dV(D_2)}{dD_2} = \left(r - 1 + D_2 \frac{dr}{dD_2}\right)\left(\frac{T}{r^\circ}\right)(1 - F((r-1)D_2)), \tag{14}$$

which is nonnegative implying that the bound in (13) is increasing in the debt level for all distribution functions and for all debt levels such that $F((r-1)D_2) < 1$. The derivative is composed of two parts, $(r-1)(T/r^\circ)$ which represents the additional tax shield from an additional unit of debt evaluated at the existing interest rate r, and $D_2 \frac{dr}{dD_2} (T/r^\circ)$ which represents the added tax shield resulting from the increase in the nominal interest rate when debt is increased. Both of these effects are multiplied by the probability that the tax protection of the interest charges will be fully utilized. Without a risk of default on the interest charges, the marginal value of debt financing is $\left(r - 1 + D_2 \frac{dr}{dD_2}\right)(T/r^\circ)$.

9. The value of the unlevered firm is assumed to be constant with respect to the debt level of the levered firm.

The bound developed in (13) is based on the assumption that all equity holders in the levered firm hold bonds of that firm. If an equity owner in the levered firm does not own bonds of that firm but may sell short, he can sell his equity and sell short $\alpha D_2$ of the bonds and obtain the return $Y^*_1$ less the covering of the short sale. The returns, $Y_2{}^S$ and $Y_1{}^S$ on the equity of the levered firm and the equity of the unlevered firm, respectively, are thus[10]

$$Y_2{}^S = \alpha\begin{cases} 0 & \text{if } X \leq X^\circ \\ (X - (r - 1)D_2)(1 - T) - D_2 & \text{if } X \geq X^\circ \end{cases} \tag{15}$$

$$Y_1{}^S = \alpha(V_2 - V_1)r^\circ \begin{cases} +\ 0 & \text{if } X \leq 0 \\ -\ \alpha TX & \text{if } 0 \leq X \leq (r - 1)D_2 \\ -\ \alpha T(r - 1)D_2 & \text{if } (r - 1)D_2 \leq X \leq X^\circ \\ +\ X(1 - T) - rD_2 & \text{if } X \geq X^\circ. \end{cases} \tag{16}$$

The distribution function for $Y_1{}^S$ is below (above) that for $Y_2{}^S$ for $X >(<) (V_2 - V_1)r^\circ/T$, and thus the distribution of $Y_1{}^S$ dominates that for $Y_2{}^S$ if and only if $\mu_2{}^S \leq \mu_1{}^S$. All investors with concave utility functions will be willing to make the transaction if $\mu_1{}^S \geq \mu_2{}^S$ since that transaction will increase their expected utility. This may be shown to imply the same bound given in (13).

Consequently, if equity owners possess bonds of the levered firm or are able to sell bonds short, the bound in (13) will hold. With the possibility of short sales, any investor may sell short an $\alpha$ proportion of the debt and equity of the levered firm and invest the proceeds in the equity of the levered firm and in savings. The return on this transaction is $Y^*_1$ in (12) less the covering of the short sales which equals $Y_1{}^S$ in (16) minus $Y_2{}^S$ in (15). All investors prefer to make this exchange until (13) is satisfied.

The same bound may be obtained by considering the equity owners of the unlevered firm. If an equity owner sells an $\alpha$ share of the equity and reduces savings by $\alpha(V_2 - V_1)$ in order to purchase an $\alpha$ share of the debt and equity of the levered firm, the gross return on the former investment is $Y^*_1$ in (12), and the return on the latter investment is $Y_2$ in (10). No equity owner of the unlevered firm will make this transaction if $\mu_2 < \mu^*_1$. Thus, when the bound in (13) is satisfied, there may be some investors who will sell equity in the unlevered firm. If all investors were risk neutral, all equity owners in the unlevered firm would make this transaction if $\mu_2 > \mu^*_1$ and all equity owners in the levered firm would make one of the opposite transactions previously described. Then, $\mu_2 = \mu^*_1$ and the bound in (13) holds as an equality. It is risk aversion that can lead to a strict inequality in (13).

To examine the effect of risk aversion in the context of the investments yielding $Y^*_1$ and $Y_2$, assume that there are J investors with homogeneous expectations, that each investor has an increasing, concave utility function $U_j$, and that the utility functions can be completely ordered in terms of their corresponding Arrow-Pratt [1, 13] indices of absolute risk aversion

10. The investor is assumed to cover the short sale obligations with other assets if necessary.

$r_j = -U''_j/U'_j$. The investors will be labeled in correspondence with their degree of risk aversion with investor 1 being the least and investor J being the most risk averse. Define $(V_2 - V_1)^\circ$ as the difference in the values of the firm such that investor 1 is exactly indifferent between $Y^*_1$ and $Y_2$, and assume that the difference in the values of the firms is initially $(V_2 - V_1)^\circ$. Since the distribution functions of $Y^*_1$ and $Y_2$ cross only once and the distribution function of $Y^*_1$ is below that for $Y_2$ to the left of the crossing point, all investors who are more risk averse than investor 1 prefer $Y^*_1$ to $Y_2$, as indicated by the strong form of theorem 3 of Hammond [6]. Consequently, at $(V_2 - V_1)^\circ$ investor 1 is indifferent between the two investments and all other investors who are equity holders in the levered firm prefer to hold their equity and savings. No investor has an incentive to sell equity in the unlevered firm and invest in the levered firm to obtain $Y_2$.

Now consider all investors in the levered firm and assume, for example, that they also hold bonds of that firm. Then, all who are more risk averse than investor 1 prefer to sell equity and bonds in the levered firm and to purchase equity in the unlevered firm and save the excess. This process will reduce the difference $(V_2 - V_1)$ to a level below that of $(V_2 - V_1)^\circ$. Investor 1 will then become an investor in the levered firm instead of in the unlevered firm, and if $V_2 - V_1$ continues to fall, investor 1 will hold his investment in the securities of the levered firm. This process will continue until a (partial) equilibrium results with all more risk averse investors holding $Y^*_1$ and all less risk averse investors holding $Y_2$. The exact difference in the values of the levered and unlevered firms cannot be determined without specifying the utility functions, but the effect of risk aversion on the relative values of the firm is clear. More risk averse investors will prefer the equity of the unlevered firm plus savings because that yields a "less risky" return, $Y^*_1$, while less risk averse investors will be willing to accept the "more risky" return $Y_2$ offered by the securities in the levered firm. In the next section, a bound on the nominal interest rate r is developed and this bound is used in sub-section F to illustrate the effect of risk aversion.

### E. *A Bound on the Nominal Interest Rate*

The bounds provided in (5), (8) and (13) are expressed in terms of the nominal interest rate on the bonds of the levered firm. While this interest rate is observable, it is desirable to relate it to the risk-free interest rate. The exact relationship between the risk-free rate of interest and the nominal rate of interest cannot be determined without more restrictive assumptions regarding the structure of the capital asset market, but a bound on the relationship may be determined using stochastic dominance for the case of homogeneous expectations. Consider a lender who may only lend an amount $\beta D_2$ at the risk-free rate $r^\circ$ or purchase $\beta D_2$ of the bonds of the levered firm. The total return $\beta R_D D_2$ on the bonds may be preferred to the return $\beta r^\circ D_2$ on the risk-free asset only if the distribution of $\beta r^\circ D_2$ does not dominate that of $\beta R_D D_2$. Clearly, $r > r^\circ$ is a necessary and sufficient

condition for the bonds not to be dominated for all increasing utility functions in the presence of default risk. If all lenders have strictly concave utility functions, then the bonds are nondominated only if $E(R_D D_2) > E(r^\circ D_2)$. The difference in the expected values is given by

$$E(R_D D_2) - E(r^\circ D_2) = \int_0^{(r-1)D_2} X dF(X) + \int_{(r-1)D_2}^{X^\circ} (X(1 - T) + T(r - 1)D_2) dF(X) + rD_2(1 - F(X^\circ)) - r^\circ D_2.$$

Integrating by parts yields

$$E(R_D D_2) - E(r^\circ D_2) = - \int_0^{(r-1)D_2} F(X)dX - \int_{(r-1)D_2}^{X^\circ} (1 - T)F(X)dX + (r - r^\circ)D_2.$$

Consequently, the bonds of the levered firm are nondominated only if

$$(r - r^\circ)D_2 > \int_0^{(r-1)D_2} F(X)dX + \int_{(r-1)D_2}^{X^\circ} (1 - T)F(X)dX. \tag{17}$$

The difference between the nominal and the risk-free rates must be at least as great as the right side of (17) divided by $D_2$. The first term on the right side reflects the possible loss to bondholders if there is default on both principal and interest, and the second term reflects the loss if there is default only on the principal. This relationship does not yield empirically testable results, since it depends on the investor's assessed distribution of gross earnings X, but it does relate the interest rates to investor expectations for all concave utility functions. It will be utilized below to relate the upper and lower bounds on the firm values.

## F. *A Lower Bound with Default Risk*

When the probability of default on debt principal and interest is positive, a lower bound on the differences in values of the levered and unlevered firms may be obtained if all investors have concave utility functions and homogeneous expectations. Using the returns in (1) and (4) and for $(V_2 - V_1)$ between the bounds in (5) and (8), the distribution functions for $Y_1$ and $Y_2$ cross only once and the distribution function of $Y_2$ is less than that for $Y_1$ to the left of the crossing point. Consequently, all investors with concave utility functions will not prefer $Y_1$ to $Y_2$, and all will prefer $Y_2$ to $Y_1$ if and only if $\mu_2 \geq \mu_1$.[11] The difference in expected values, after integrating by parts, yields the bound

$$V_2 - V_1 \geq T(r - 1)D_2/r + \frac{T\left((1 - T)(r - 1) \int_{(r-1)D_2}^{X^\circ} F(X)dX - \int_0^{(r-1)D_2} F(X)dX\right)}{r(1 - T)E(X)/V_1}. \tag{18}$$

The first term on the right side is the same as the term in (5), and the second-term is positive reflecting an additional value resulting from a level of debt that yields a default risk. The second term may be inter-

11. If there is a cost C incurred when the firm becomes insolvent, then the return is zero for $0 \leq X \leq C$. It is then possible for the value of the levered firm to be less than that of the unlevered firm. This might occur if investors were very risk averse, for example. Kraus and Litzenberger [8] have considered such an insolvency cost in the context of a state preference model.

preted as a value of debt financing when there is default risk with the first integral reflecting the increase in value when there is default risk on the debt principal and the second integral reflecting a decrease in value due to the probability that the tax shield will not be fully utilized. These two effects are relative to the expected return $[(1 - T)E(X)/V_1]$ per dollar invested in the unlevered firm.[12] The right side of (18) is a strictly increasing function of the debt level reflecting the tax shield provided by the interest charges as well as the increase in the nominal interest rate when there is a probability of default.

To explore the relationship between the lower bound in (18) and the upper bound in (13), assume that all investors are risk neutral. Then the bounds hold as equalities and the expected return on all investments equals the risk-free interest rate, so $(1 - T)E(X)/V_1 = r_o$. The equalities in (18) and (13) together imply

$$\begin{aligned} 0 &= \frac{T(r-1)D_2}{r} + \frac{T\left((1-T)(r-1)\int_{(r-1)D_2}^{X^o} F(X)dX - \int_0^{(r-1)D_2} F(X)dX\right)}{rr^o} \\ &\quad - \frac{T(r-1)D_2}{r^o} + \frac{T}{r^o}\int_0^{(r-1)D_2} F(X)dX \qquad (19) \\ &= \frac{T(r-1)}{rr^o}\left((r^o - r)D_2 + (1-T)\int_{(r-1)D_2}^{X^o} F(X)dX - \int_0^{(r-1)D_2} F(X)dX\right). \end{aligned}$$

This is a constant $(T(r-1)/rr^o)$ multiplied by the equilibrium relationship between the risk-free and the nominal interest rate in (19), which is zero if all investors are risk neutral. The upper bound in (13) and the lower bound in (18) are thus identical in the case of risk neutrality but can be different if some investors are risk averse.

## III. Summary

A number of bounds on the value of debt financing have been developed in a partial equilibrium context, and these bounds will be summarized with the aid of Figure 1. If the debt level is such that there is no risk of default $(D_2 \leq D^*)$[13] on the principal and interest charges, the value of debt financing is given by $T(r^o - 1)D_2/r^o$. This is represented by the solid line over the interval $[0,D^*]$. For debt levels greater than $D^*$, the nominal interest rate increases, and the value per dollar of debt financing is greater than $T(r^o - 1)D_2/r^o$ which is represented by the dashed line. The value of debt financing is greater than $T(r - 1)D_2/r$ given in (3) (the dotted line in Figure 1) and is exactly given by

12. The same bound may be obtained by considering the sales of the securities of the levered firm and the purchase of the equity of the unlevered firm. For example, no investor with a concave utility function who holds $\alpha kD_2$ of the bonds and $\alpha E_2$ of the levered firm will sell those securities and purchase equity of the unlevered firm unless (18) is satisfied. Similarly, no investor with a concave utility function will prefer to sell short $\alpha E_2$ of the equity of the levered firm and purchase $\alpha Q_2$ of the equity of the unlevered firm unless (18) is satisfied.

13. If $X_{min}$ denotes the minimum possible earnings, the debt level $D^*$ is defined by $D^* - \max\{0, X_{min}/(1/(1-T) + r^o - 1)\}$.

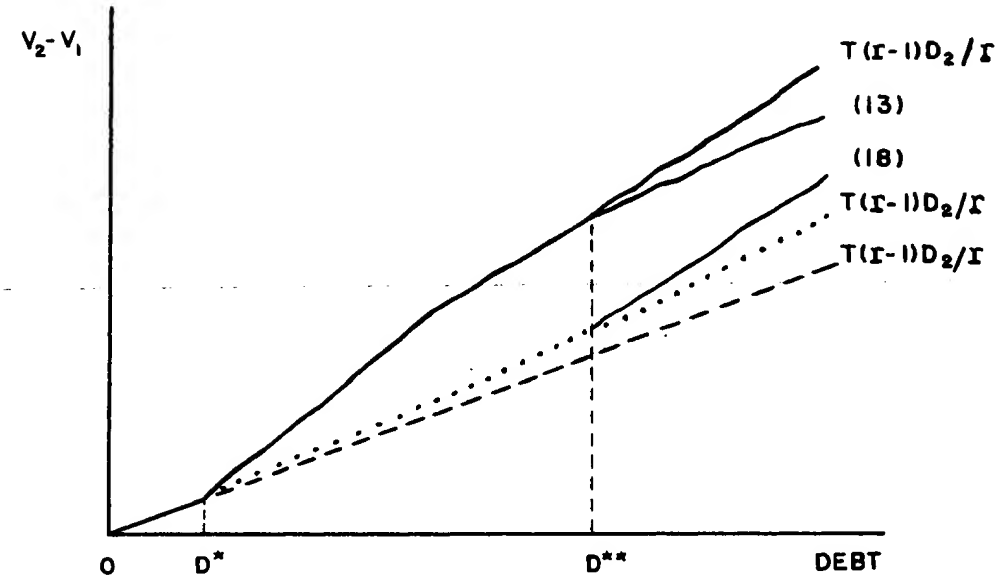


FIGURE 1

$T(r - 1)D_2/r^\circ$ in (8) for debt levels such that there is no risk of default on the interest payments ($D_2 \leq D^{**}$). For debt levels beyond $D^{**}$ the value of debt financing is bounded above by the relationship in (13) and bounded below by the relationship in (18) as indicated in Figure 1. The upper bound is less than $T(r - 1)D_2/r^\circ$ because there is a risk that the tax protection of the interest charges may not be fully utilized, but the value of additional debt financing is still positive. These bounds are rather tight and with the assumption of risk neutrality for all investors, they are identical. Risk aversion may however yield a value of debt financing that is between the bounds given in (13) and (18).

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